

[This question paper contains 4 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 4792

E

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : B.Sc. (H) Mathematics (CBCS – LOCF)

Semester : VI

Duration : 3 Hours

Maximum Marks : 75

**Instructions for Candidates**

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the questions are compulsory.
3. Attempt any two parts from each question.
4. Marks of each part are indicated

1. (a) (i) Prove that If  $F$  is a field, then  $F[x]$  is a Principal Ideal Domain.

(ii) Is  $\mathbb{Z}[x]$ , a Principal Ideal Domain? Justify your answer.

(b) Prove that  $\langle x^2 + 1 \rangle$  is not a maximal ideal in  $\mathbb{Z}[x]$ .

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- (c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)
2. (a) (i) State and prove Gauss's Lemma.  
 (ii) Is every irreducible polynomial over  $\mathbb{Z}$  primitive? Justify.
- (b) Construct a field of order 25.
- (c) In  $\mathbb{Z}[\sqrt{-5}]$ , prove that  $1 + 3\sqrt{-5}$  is irreducible but not prime. (4+2.5,6.5,6.5)
3. (a) Let  $V = \mathbb{R}^3$  and define  $f_1, f_2, f_3 \in V^*$  as follows:  
 $f_1(x, y, z) = x - 2y,$   
 $f_2(x, y, z) = x + y + z,$   
 $f_3(x, y, z) = y - 3z.$   
 Prove that  $\{f_1, f_2, f_3\}$  is a basis for  $V^*$  and then find a basis for  $V$  for which it is the dual basis.
- (b) Test the linear operator  $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), T(f(x)) = f(0) + f(1)(x + x^2)$  for diagonalizability and if diagonalizable, find a basis  $\beta$  for  $V$  such that  $[T]_\beta$  is a diagonal matrix.
- (c) Let  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$ . Find an expression for  $A^n$ , where  $n$  is an arbitrary natural number. (6,6,6)

4. (a) For a linear operator  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $T(a, b, c) = (-b + c, a + c, 3c)$ , determine the  $T$ -cyclic subspace  $W$  of  $\mathbb{R}^3$  generated by  $e_1 = (1, 0, 0)$ . Also find the characteristic polynomial of the operator  $T_W$ .
- (b) State Cayley-Hamilton theorem and verify it for the linear operator  $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ ,  $T(f(x)) = f'(x)$ .
- (c) Show that the vector space  $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$  where  $W_1 = \{(a, b, 0, 0) : a, b \in \mathbb{R}\}$ ,  $W_2 = \{(0, 0, c, 0) : c \in \mathbb{R}\}$  and  $W_3 = \{(0, 0, 0, d) : d \in \mathbb{R}\}$ .  
(6.5, 6.5, 6.5)
5. (a) Consider the vector space  $\mathbb{C}$  over  $\mathbb{R}$  with an inner product  $\langle \cdot, \cdot \rangle$ . Let  $\bar{z}$  denote the conjugate of  $z$ . Show that  $\langle \cdot, \cdot \rangle'$  defined by  $\langle z, w \rangle' = \langle \bar{z}, \bar{w} \rangle$  for all  $z, w \in \mathbb{C}$  is also an inner product on  $\mathbb{C}$ . Is  $\langle \cdot, \cdot \rangle''$  defined by  $\langle z, w \rangle'' = \langle z + \bar{z}, w + \bar{w} \rangle$  for all  $z, w \in \mathbb{C}$  an inner product on  $\mathbb{C}$ ? Justify your answer.
- (b) Let  $V = P(\mathbb{R})$  with the inner product  $\langle p(x), q(x) \rangle = \int_{-1}^1 p(t)q(t)dt \quad \forall p(x), q(x) \in V$ . Compute the orthogonal projection of the vector  $p(x) = x^{2k-1}$  on  $P_2(\mathbb{R})$ , where  $k \in \mathbb{N}$ .

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(c) (i) For the inner product space  $V = P_1(\mathbb{R})$  with  $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$  and the linear operator  $T$  on  $V$  defined by  $T(f) = f' + 3f$ , compute  $T^*(4 - 2t)$ .

(ii) For the standard inner product space  $V = \mathbb{R}^3$  and a linear transformation  $g: V \rightarrow \mathbb{R}$  given by  $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$ , find a vector  $y \in V$  such that  $g(x) = \langle x, y \rangle$  for all  $x \in V$ .  
(6,6,2+4)

6. (a) Prove that a normal operator  $T$  on a finite-dimensional complex inner product space  $V$  yields an orthonormal basis for  $V$  consisting of eigenvectors of  $T$ . Justify the validity of the conclusion of this result if  $V$  is a finite-dimensional real inner product space.

(b) Let  $V = M_{2 \times 2}(\mathbb{R})$  and  $T: V \rightarrow V$  be a linear operator given by  $T(A) = A^T$ . Determine whether  $T$  is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of  $T$  for  $V$  and list the corresponding eigenvalues.

(c) For the matrix  $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$  find an orthogonal

matrix  $P$  and a diagonal matrix  $D$  such that  $P^*AP = D$ .  
(6.5,6.5,6.5)